Numerical Scheme for the Neutron Transport Equation with the Method of Characteristics

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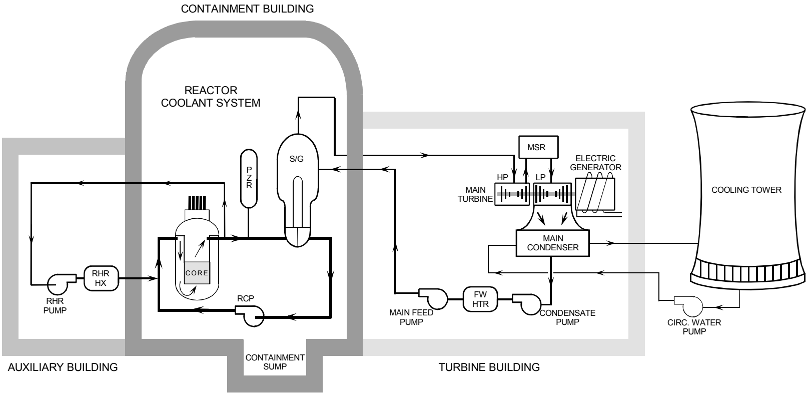
Outline

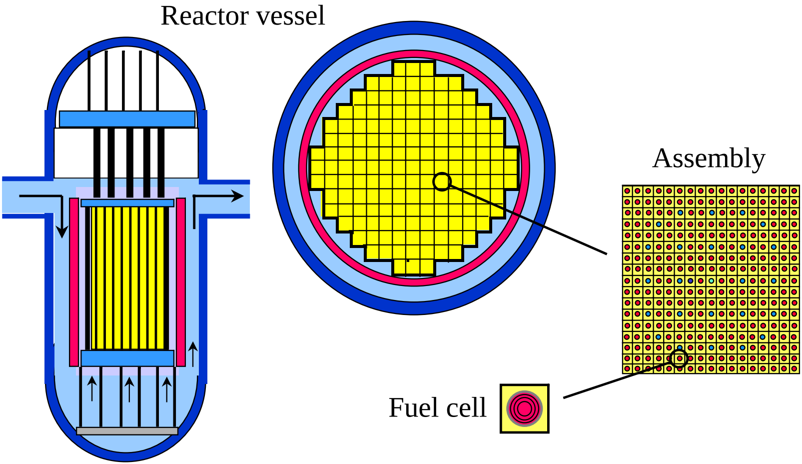
Pressurized Water Reactor

General Overview



Pressurized Water Reactor

Core Geometry



Neutron Transport

Dependant Variables

- Neutron density: $N(\mathbf{r}, \mathbf{\Omega}, E, t) \quad [cm^{-3}]$
  number of neutrons per unit volume.

- Neutron flux: $\psi(\mathbf{r}, \mathbf{\Omega}, E, t) \quad [cm^{-2}.s^{-1}]$ $\qquad \psi = v\,N$
  number of neutrons crossing a surface element orthogonal to direction $\mathbf{\Omega}$ per unit time.

- 7 variables:
  - $\mathbf{r}$: position (3 coord.)
  - $\mathbf{\Omega}$: direction of flight (2 coord.)
  - E: energy (or speed: $E = \frac{1}{2}\,m\,v^2$)
  - t: time

Neutron Transport

Boltzmann Equation

$$\underbrace{\frac{dN}{dt}}_{accumulation} =$$

$$= 0$$

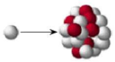
- N: neutron density

Neutron Transport

Boltzmann Equation

$$\underbrace{\frac{dN}{dt}}_{accumulation} = \underbrace{-v\,\mathbf{\Omega}\cdot\nabla_{\mathbf{r}}N}_{transport}$$

$= 0$



- N: neutron density
- v: velocity of the neutrons
- $\mathbf{\Omega}$: direction of flight

Neutron Transport

Boltzmann Equation

$$\underbrace{\frac{dN}{dt}}_{accumulation} = \underbrace{-v\,\mathbf{\Omega}\cdot\nabla_{\mathbf{r}} N}_{transport} \;\underbrace{-v\,\Sigma\,N}_{interactions}$$

$= 0$

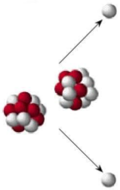
- N: neutron density
- v: velocity of the neutrons
- $\mathbf{\Omega}$: direction of flight
- Σ: cross-section (probability of interaction)

Neutron Transport

Boltzmann Equation

$$\underbrace{\frac{dN}{dt}}_{accumulation = 0} = \underbrace{-v\,\boldsymbol{\Omega}\cdot\nabla_{\mathbf{r}}N}_{transport}\ \underbrace{-v\,\Sigma\,N}_{interactions} + \underbrace{S}_{sources}$$

sources
- *scattering*
- *fission*
- *(external)*



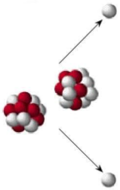
- N: neutron density
- v: velocity of the neutrons
- $\boldsymbol{\Omega}$: direction of flight
- Σ: cross-section (probability of interaction)

Neutron Transport

Boltzmann Equation

$$\underbrace{\frac{1}{v}\frac{d\psi}{dt}}_{accumulation \; = 0} = \underbrace{-\mathbf{\Omega}\cdot\nabla_{\mathbf{r}}\psi}_{transport} \; \underbrace{-\Sigma\,\psi}_{interactions} + \underbrace{S}_{sources}$$

- *scattering*
- *fission*
- *(external)*



- ψ: neutron flux ($\psi = v\,N$)
- v: velocity of the neutrons
- $\mathbf{\Omega}$: direction of flight
- Σ: cross-section (probability of interaction)

Outline

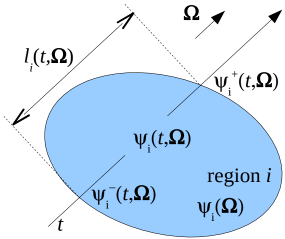
The Method of Characteristics

Classical Approach

- Assumption: homogeneous regions

$$\begin{cases} \Sigma(\mathbf{r}) = \Sigma_i, \\ q(\mathbf{r}) = q_i, \end{cases} \qquad \mathbf{r} \in \text{region } i$$

- Integrating the Boltzmann Equation over a line segment intersecting a region yields:

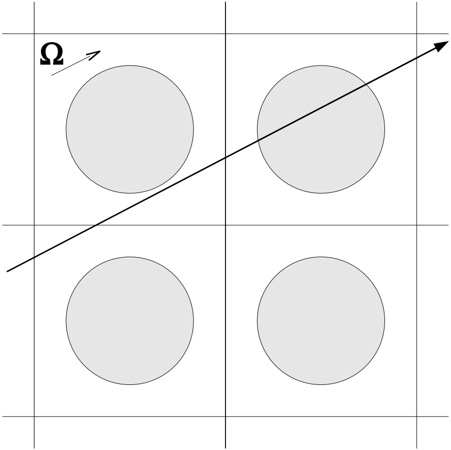


$$\psi_i^+(t,\mathbf{\Omega}) = \psi_i^-(t,\mathbf{\Omega}) \underbrace{e^{-\Sigma_i\, l_i(t,\mathbf{\Omega})}}_{\substack{transmission \\ coefficient}} + sources$$

The Method of Characteristics

Classical Approach

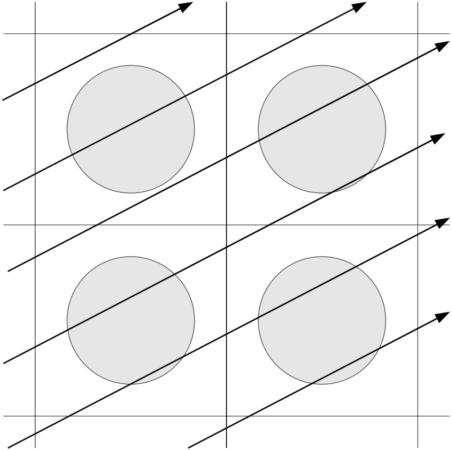
- Trajectories tracked through the whole domain



The Method of Characteristics

Classical Approach

- Trajectories tracked through the whole domain
- Several trajectories to cover the transverse extent

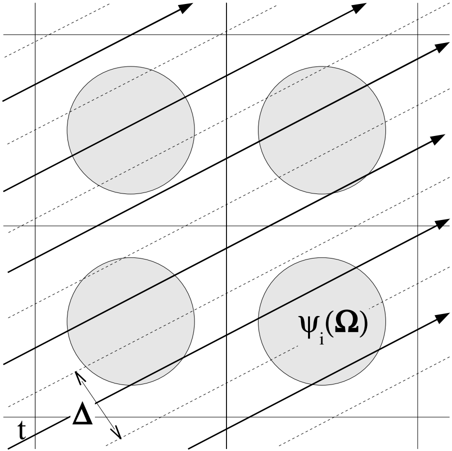


The Method of Characteristics

Classical Approach

- Trajectories tracked through the whole domain
- Several trajectories to cover the transverse extent
- Transverse integration of the average flux in a region:

$$\psi_i(\boldsymbol{\Omega}) = \frac{\displaystyle\sum_{t \cap i} \Delta\ l_i(t, \boldsymbol{\Omega})\ \psi_i(t, \boldsymbol{\Omega})}{\displaystyle\sum_{t \cap i} \Delta\ l_i(t, \boldsymbol{\Omega})}$$

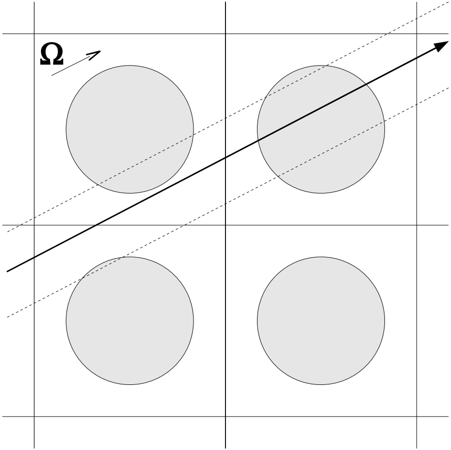


The Method of Characteristics

Tracking and Discretization Problems

Approximations due to the tracking:

- Transverse variation of the angular flux
- Transverse variation of the intersection length
- Material discontinuities

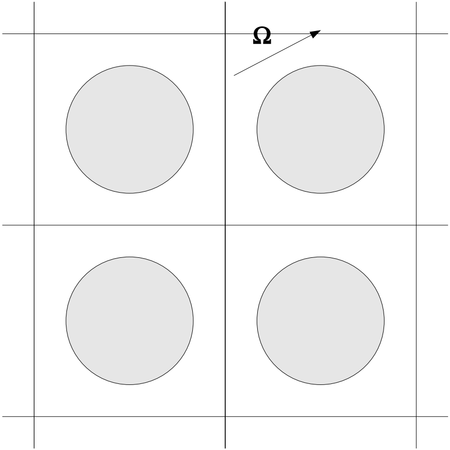


Outline

The Macroband Method

Avoiding Material Discontinuities

Direct projection of all discontinuities:

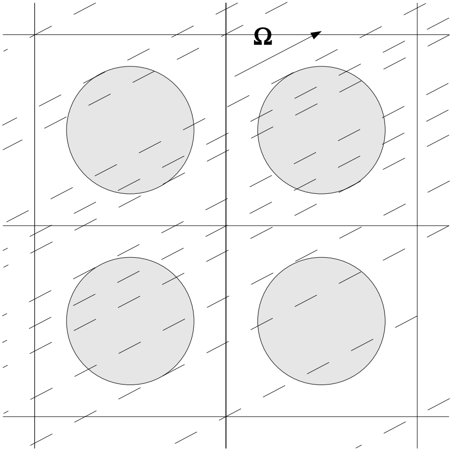


The Macroband Method

Avoiding Material Discontinuities

Direct projection of all discontinuities:

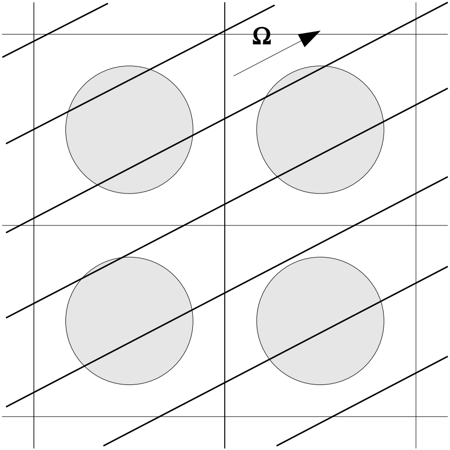
- Large number of transverse mesh cells
- Too onerous



The Macroband Method

Avoiding Material Discontinuities

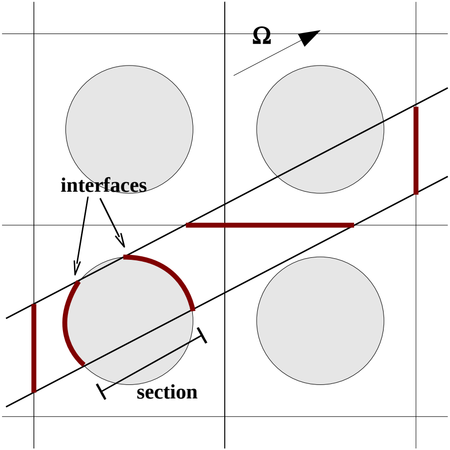
1. Define a constant-step transverse mesh



The Macroband Method

Avoiding Material Discontinuities

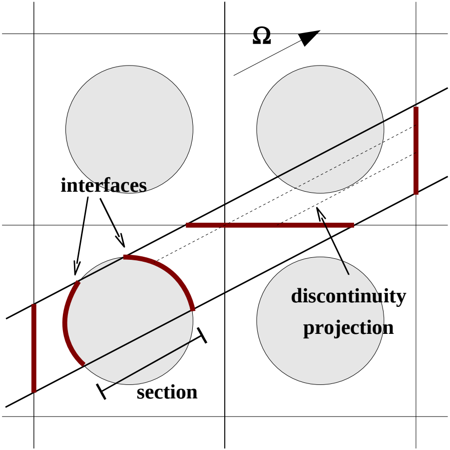
1. Define a constant-step transverse mesh
2. Split each band into sections



The Macroband Method

Avoiding Material Discontinuities

1. Define a constant-step transverse mesh
2. Split each band into sections
3. Project the discontinuities section-wise



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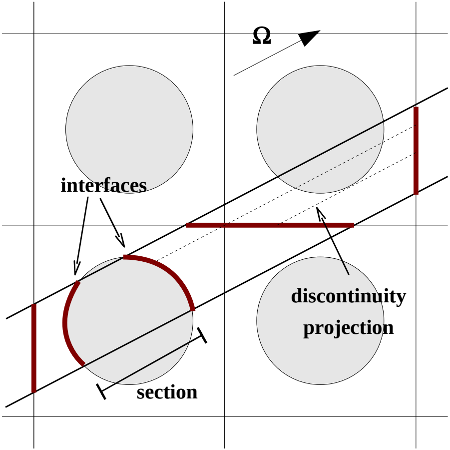
Avoiding Material Discontinuities

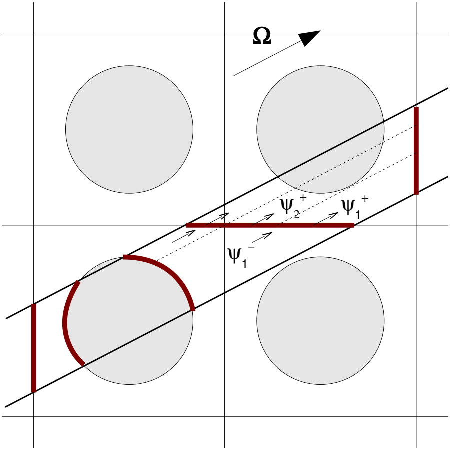
1. Define a constant-step transverse mesh
2. Split each band into sections
3. Project the discontinuities section-wise
4. Propagate the flux across each sub-band



The Macroband Method

Avoiding Material Discontinuities

1. Define a constant-step transverse mesh
2. Split each band into sections
3. Project the discontinuities section-wise
4. Propagate the flux across each sub-band
5. Redistribute the flux at section interfaces

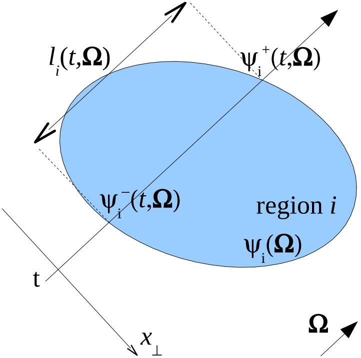


The Macroband Method

Transverse Integration Formula

- Transmission equation:

$$\psi_i^+(t,\mathbf{\Omega}) = e^{-\Sigma_i\, l_i(t,\mathbf{\Omega})}\ \psi_i^-(t,\mathbf{\Omega}) + sources$$



The Macroband Method

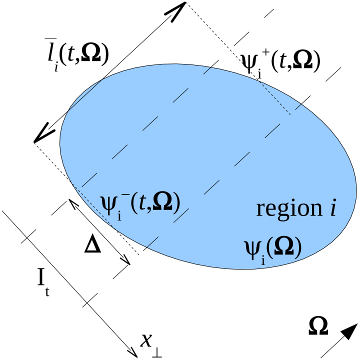
Transverse Integration Formula

- Transmission equation:

$$\psi_i^+(t,\mathbf{\Omega}) = T_i(t,\mathbf{\Omega})\ \psi_i^-(t,\mathbf{\Omega}) + sources$$

- Transverse averaged transmission:

$$T_i(t,\mathbf{\Omega}) = \frac{1}{\Delta}\int_{I_t} e^{-\Sigma_i\, l_i(x_\perp,\mathbf{\Omega})}\ dx_\perp$$



The Macroband Method

Transverse Integration Formula

- Transmission equation:

$$\psi_i^+(t,\mathbf{\Omega}) = T_i(t,\mathbf{\Omega})\ \psi_i^-(t,\mathbf{\Omega}) + sources$$

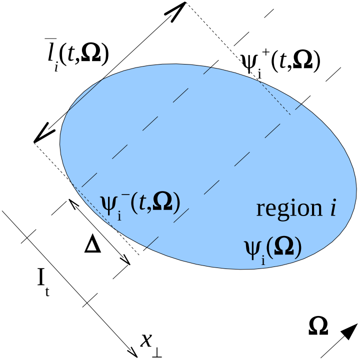
- Transverse averaged transmission:

$$T_i(t,\mathbf{\Omega}) = \frac{1}{\Delta}\int_{I_t} e^{-\Sigma_i\ l_i(x_\perp,\mathbf{\Omega})}\ dx_\perp$$

- Taylor expansion of the exponential term:

$$T_i(t,\mathbf{\Omega}) \quad \simeq \quad e^{-\Sigma_i\ \bar{l}_i(t,\mathbf{\Omega})} \sum_{p=1}^{n_k} \alpha_p \Sigma_i^p$$

$$\alpha_p \quad = \quad \frac{(-1)^p}{\Delta\ p!} \int_{I_t} \left[l_i(x_\perp) - \bar{l}_i(x_\perp)\right]^p\ dx_\perp$$



Outline

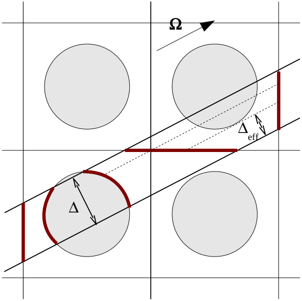
Numerical Results

Comparison between the “classical” MOC and the macroband method

- Reference calculation:
  “classical” MOC with $\Delta = 5.10^{-4}$ cm
- Effective tracking step for the macrobands:

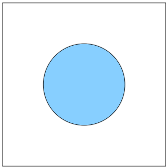
$$\Delta_{eff} = \frac{\Delta}{n_{sb}}$$

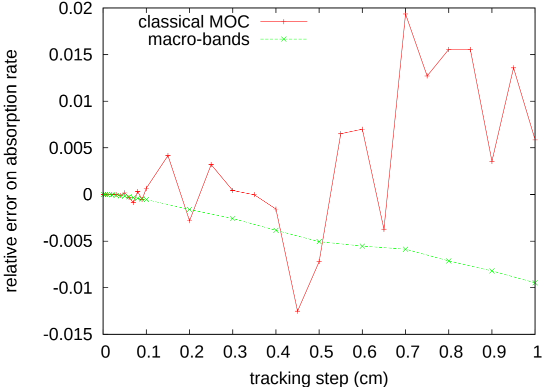
n_{sb}: average number of sub-bands per section.



Numerical Results

Convergence – Accuracy

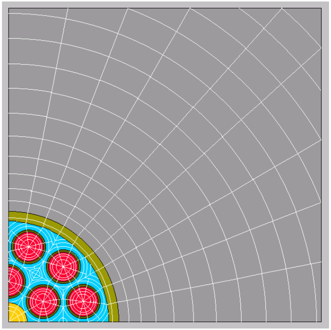


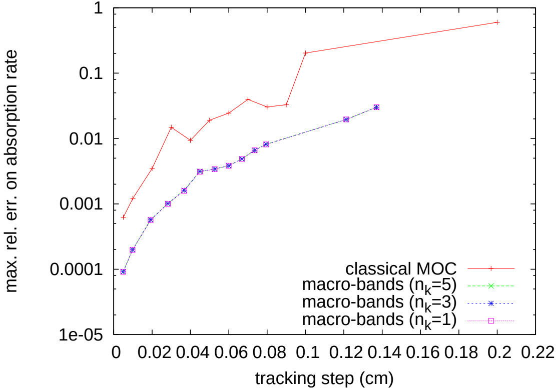


Comparison between the "classical" MOC and the macroband technique
simplified PWR Fuel Cell

Numerical Results

Accuracy

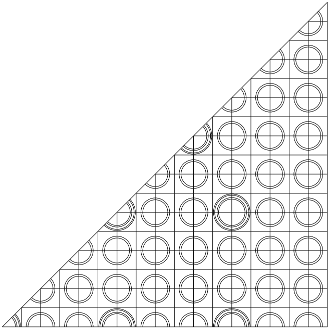


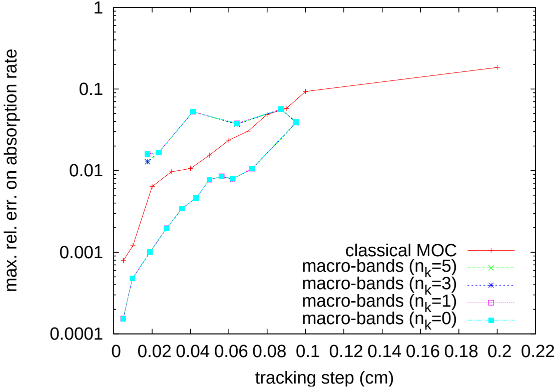


Comparison between the “classical” MOC and the macroband technique
RBMK Cell

Numerical Results

Accuracy





Comparison between the “classical” MOC and the macroband technique
PWR Rodded Assembly

Outline

Conclusions

- Numerical results obtained with the macroband method:
  - for the same precision: tracking step up to $6\times$ larger;
  - for the same number of bands: $2-3\times$ more operations;
  - up to $30-50\%$ gains in terms of computing time.
- Accuracy of the MOC
  - Gain in precision with the macroband method
    $\Rightarrow$ accuracy of the MOC limited by the transverse integration.
  - Negligible impact of the Taylor expansion order (n_k)
    $\Rightarrow$ main cause of error: material discontinuities.

Perspectives

- Optimize the macroband method implementation
- Implement an acceleration scheme for the macroband
- Implement cycling tracking for closed domains

- Use a piecewise linear transverse expansion for the flux

Thank you for your attention

Bibliography

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Numerical Results

Computing Time

- Comparison of algorithmic complexities:

tracking technique	exponentials	products	additions	tracking storage
classical MOC	1	2	2	1
Macrobands with Taylor expansion	1	$3 + n_k + r$	$2 + n_k$	$1 + n_k + 2r$
Macrobands only	1	$2 + r$	2	$1 + 2r$

- r: average cost for flux repartition at interfaces

$$r \simeq 0.6$$

The Method of Characteristics

Classical Approach

- Transport equation in a geometric domain D:

$$\begin{cases} (\mathbf{\Omega} \cdot \nabla_{\mathbf{r}} + \Sigma)\psi = q, & (\mathbf{r}, \mathbf{\Omega}) \in D \times S_N \\ \psi = \psi_0 + \beta\, \psi, & (\mathbf{r}, \mathbf{\Omega}) \in \partial D \times S_{N_-} \end{cases}$$

- D is composed of unstructured homogeneous regions:

$$\begin{cases} \Sigma(\mathbf{r}) = \Sigma_i, \\ q(\mathbf{r}) = q_i, \end{cases} \qquad \mathbf{r} \in \text{region } i$$

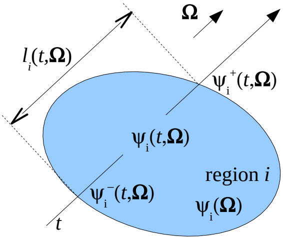
The Method of Characteristics

Classical Approach

Integrating the Boltzmann Equation over a line segment intersecting a region yields:

- Transmission equation:

$$\psi_i^+(t,\mathbf{\Omega}) = \psi_i^-(t,\mathbf{\Omega})\, e^{-\Sigma_i\, l_i(t,\mathbf{\Omega})} + \frac{1-e^{-\Sigma_i\, l_i(t,\mathbf{\Omega})}}{\Sigma_i}\; q_i(\mathbf{\Omega})$$



- Balance equation:

$$\psi_i(t,\mathbf{\Omega}) = \frac{q_i(\mathbf{\Omega})}{\Sigma_i} + \frac{\psi_i^-(t,\mathbf{\Omega}) - \psi_i^+(t,\mathbf{\Omega})}{\Sigma_i\, l_i(t,\mathbf{\Omega})}$$

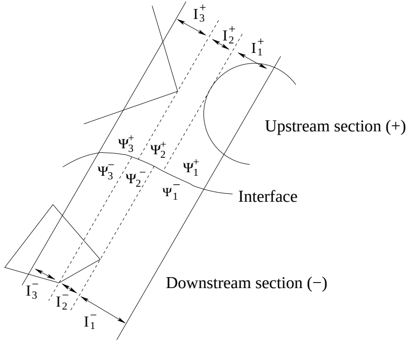
The Macroband Method

Avoiding Material Discontinuities

Flux redistribution at section interfaces:

- Preserve the currents
- Flat flux assumption in each sub-band
- $$\psi_k^+ = \sum_{k\prime} \frac{\Delta_{k,k\prime}}{\Delta_k} \psi_{k'}^-$$

$$\Delta_k = l(I_k^+)$$
$$\Delta_{k,k'} = l(I_k^+ \cap I_{k'}^-)$$



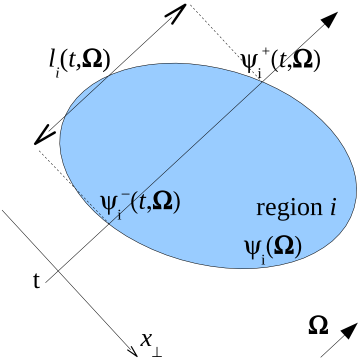
The Macroband Method

Transverse Integration Formula

- Transverse averaged transmission:

$$\psi_i^+(t,\mathbf{\Omega}) = \quad e^{-\Sigma_i\, l_i(t,\mathbf{\Omega})}\ \psi_i^-(t,\mathbf{\Omega})$$

$$+\frac{1-e^{-\Sigma_i\, l_i(t,\mathbf{\Omega})}}{\Sigma_i}\ q_i(\mathbf{\Omega})$$



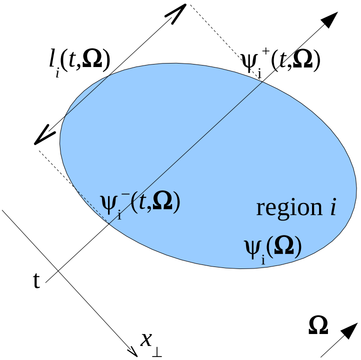
The Macroband Method

Transverse Integration Formula

- Transverse averaged transmission:

$$\psi_i^+(t,\mathbf{\Omega}) = T_i(t,\mathbf{\Omega})\, \psi_i^-(t,\mathbf{\Omega}) + \frac{1 - T_i(t,\mathbf{\Omega})}{\Sigma_i}\, q_i(\mathbf{\Omega})$$

$$T_i(t,\mathbf{\Omega}) = \frac{1}{\Delta} \int_{I_t} e^{-\Sigma_i\, l_i(x_\perp,\mathbf{\Omega})}\, dx_\perp$$



The Macroband Method

Transverse Integration Formula

- Taylor expansion for the exponential term:

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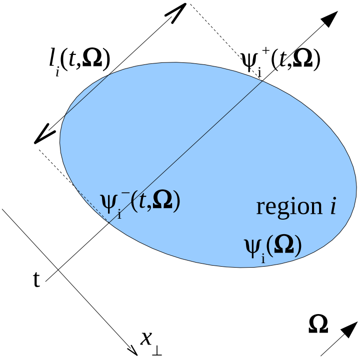

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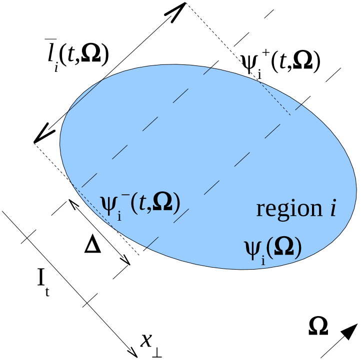
$$T_i(t,\mathbf{\Omega}) \quad = \quad \frac{1}{\Delta}\int_{I_t} e^{-\Sigma_i\, l_i(x_\perp,\mathbf{\Omega})}\; dx_\perp$$

$$\simeq \quad e^{-\Sigma_i\, \bar{l}_i(t,\mathbf{\Omega})} \sum_{p=1}^{n_k} \alpha_p \Sigma_i^p$$

$$\alpha_p = \frac{(-1)^p}{\Delta\; p!} \int_{I_t} \left[l_i(x_\perp) - \bar{l}_i(x_\perp)\right]^p \; dx_\perp$$



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